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Critical Exponents from the Resummed Effective Potential of the $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ Scalar Field Theory

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We establish a unified way for the calculation of the critical exponents, without the use of epsilon expansion, through the improvement of the perturbative effective potential of the 1+1 dimensional $(\frac{g}{4}\phi^4 - J\phi)$ scalar field theory. First, we obtain the perturbation series for the effective potential up to g^3. We improved the perturbative effective potential by establishing a parameter-free resummation algorithm, originally due to Kleinert, Thoms and Janke, which has the privilege of using the strong coupling as well as the large coupling behaviors rather than the conventional resummation techniques which use only the large order behavior. Accordingly, although the perturbation series available is up to g^3 order, we found a complete agreement between our resummed effective potential and the well known features from constructive field theory. We prove that the 1-PI correlation functions and the effective potential ought to have the same large order as well as strong coupling behaviors. We computed the critical exponents and our results show a good agreement with the exact Ising model values.

KEY WORDS: effective potential; critical phenomena; exponents; Borel resummation.

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1. INTRODUCTION

The usual recipe of the calculation of the critical exponents is to obtain different amplitudes, perturbatively, and then apply a resummation technique to obtain reliable results. This way, although of it's success, is time consuming as one has to calculate different types of Feynman diagrams. Another route is to use the equation of state or a master formula that can produce all the amplitudes through mathematical operations. This route saves the effort as one needs to calculate the

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Feynman diagrams for only one amplitude. The master formula we mean is the vacuum energy which plays the role of the generating functional for all the n-point functions (1pI correlation functions) (Peskin and Schroeder, 1995).

To go through, we mention some other successful attempts (time consuming) to study the critical phenomena in quantum field theories. Pablo J. Marrero *et al.* (1999) used the Monte Carlo simulation of the diffusion equation in the spherical wave expansion and found a good agreement with the universality class predictions (Zinn-Justin, 1993) for the order of the phase transition and the critical exponents. Also, in a series of articles Le Guilliou *et al.* (Brezin *et al.*, 1980; Le Guillou and Zinn-Justin, 1980; Zinn-Justin, 1993) performed a Borel resummation of the perturbation series for the renormalization group functions. Their first results, obtained by the perturbative expansion up to the order g^4 as a starting point, did not give critical exponents close to those prescribed by Ising model. Subsequently, Le Guilliou *et al.* presented the results of the Borel summation of ϵ-expansion series for the renormalization group functions obtained in Kazakov *et al.* (1979) with a homographic transformation of the form $\tilde{\epsilon} = \frac{\epsilon}{1-\frac{\epsilon}{3}}$. The result was further improved by applying a conformal transformation

$$\tilde{E}(G) = E\left(\frac{G}{1-\tau G}\right), \tag{1}$$

where the Borel resummation was performed for the series $\tilde{E}(G)$ instead of $E(G)$ (Zinn-Justin, 1993) and G is the dimensionless coupling constant. Using these modifications, Le Guilliou *et al.* were able to obtain $\beta = 0.12$, $\gamma = 1.73$ and $\nu = 0.99$ in a good agreement with two-dimensional Ising model ($\beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ and $\nu = 1$).

While such results themselves are extremely interesting, useful and can be rather immediately tested, one may be also interested in having a more unified description of the model's behavior including the regions away from the critical region. We stress that the calculations using ϵ-expansion include the calculations of many amplitudes with many different types of Feynman diagrams. Also, the conventional Borel algorithm used in Brezin *et al.* (1980); Le Guillou and Zinn-Justin (1980); Zinn-Justin (1993) has some unfixed parameters while it is possible, as we will see in this work, to fix all the parameters by employing all the known asymptotic behaviors of the resummed series.

The more coherent description including the region away from the critical point may be provided by an effective potential. Detailed knowledge of an effective potential may also offer physically clear insight about the properties of field-theoretic model and the detailed dynamics of symmetry rearrangement during the phase transition. Moreover, the use of the effective potential saves the effort of calculations in a sense that we need only to calculate one kind of Feynman diagrams, namely, the vacuum diagrams. In this work, we intend to carry out calculations of the critical exponents from the effective potential for $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ theory using perturbative expansion supplemented by a Borel resummation. We attempt to offer a coherent description of the model by computing perturbative effective potential away from the critical region in the Symmetric (S) phase for small coupling G (the coupling G is to be defined later) and in the Broken Symmetry (BS) phase for large coupling G. We then attempt to approach the critical region. This will enable us, in principle, to extract all the critical quantities from only one master formula.

To shed light on previous tries for the study of the critical phenomena in the regime of the effective potential, we assert that the Gaussian Effective Potential (GEP) (Stevenson, 1985) and the Hartree Approximation (HA) (Chang and Wright, 1975; Chang and Yan, 1975), Chang's method (Chang, 1976; Magruder, 1976), the Oscillator Representation method (Dineykhan *et al.*, 1995; Efimov, 1989) and the perturbative effective potential shared the similar errors regarding the order of the phase transition in $\frac{g}{4}\phi^4_{1+1}$ scalar field theory. Among these approaches, the direct calculation of the effective potential stands out due to it's capability to employ other non-perturbative tools like Borel resummation. However, the conventional Borel algorithms needs a relatively high order in perturbation series while, up to the best of our knowledge, the available series for the effective potential for the $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ theory is up to two loops (Cea and Tedesco, 1994) (for $J = 0$ only). As a part of this work, we will obtain the series up to g^3 order in order to have richer input information to make the Borel resummation results more reliable.

Unlike the other variational techniques like GEP, the effective potential can attain perturbative corrections in a systematic way. Still, as indicated above, the effective potential of $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ theory yields the first-order phase transition inconsistent both with the universality argument and the constructive field theory (Simon, 1974; Simon and Griffiths, 1973) due to persistence of the phase transition under the small external perturbations. Nonetheless, by taking the first order effective potential for the ground state energy of $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$field theory as a starting point, one may hope to improve it in the near-critical region by accounting explicitly for higher order perturbative corrections. Since the effective coupling goes like $\frac{G}{t}$, these corrections are small both for small as well as large G. However, at the critical region, the corrections are very large and thus a non-perturbative tool should follow the perturbation calculations to get reliable results concerning the critical phenomena.

In this work, we employ the Borel resummation as a non-perturbative tool to approach the critical region. As mentioned above, the Borel resummation is indispensable for the better description of the critical region where the effective interaction in both phases is not weak. Thus, we will offer a description of the original model in terms of an effective potential which we think is reliable for all values of the dimensionless coupling. We will show that the second order phase transition is obtained in such treatment for $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ in agreement with the universality arguments. Also, we will show that all the n-point functions as well as the effective potential have the same Borel parameters. The most interesting result in this work is the calculation of the critical exponents from the effective potential itself. A note to be mentioned is that the Borel algorithm we use here uses one more piece of information than the algorithm used in Zinn-Justin (1993), namely, the strong coupling behavior of the series to be summed. Accordingly, the algorithm we use is free of parameters as well as resulting in reliable calculations with a relatively low order in perturbation series.

The paper is organized as follows. In Section 2, we briefly review the first order effective potential using normal ordering. In Section 3, we obtain the perturbation series for the effective potential up to g^3. In this section, also, we summarize the key points of the Borel-Summation with Kleinert algorithm. We discuss the results in Section 4. Conclusion follows in Section 5.

2. THE NORMAL ORDERED EFFECTIVE POTENTIAL

In low dimensional super-renormalizable theories, it is often enough to work with normal ordering to render the quantum field theory finite. This is because there are only few diagrams that are divergent and these are regulated by normal ordering. The $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ theory is such an example that has only one divergent diagram in the self-energy amplitude. In that case, one shall start with a Hamiltonian that is normal ordered with respect to the vacuum of mass parameter m;

$$H = N_m\left(\frac{1}{2}((\nabla\phi)^2 + \pi^2 + m^2\phi^2) + \frac{g}{4}\phi^4 - J\phi\right), \tag{2}$$

where π is the conjugate momentum of the field ϕ. We can use the relation (Coleman, 1975)

$$N_m \exp(i\beta\phi) = \exp\left(-\frac{1}{2}\beta^2\Delta\right) N_{M=\sqrt{t}\cdot m} \exp(i\beta\phi), \tag{3}$$

to rewrite the Hamiltonian normal ordered with respect to a new mass parameter $M = \sqrt{t}\cdot m$. In Eq. (3), expanding both sides and equating the coefficients of the same power in β yields the result

$$\begin{aligned} N_m\phi &= N_M\phi, \\ N_m\phi^2 &= N_M^2\phi^2 + \Delta, \\ N_m\phi^3 &= N_M\phi^3 + 3\Delta N_M\phi, \\ N_m\phi^4 &= N_M\phi^4 + 6\Delta N_M\phi^2 + 3\Delta^2, \end{aligned} \tag{4}$$

with

$$\Delta = -\frac{1}{4\pi}\ln t. \tag{5}$$

Also, it is easy to obtain the result (Dineykhan *et al.*, 1995)

$$N_m\left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\pi^2\right) = N_M\left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\pi^2\right) + \frac{1}{8\pi}(M^2 - m^2). \tag{6}$$

Then we apply the canonical transformation (Dineykhan *et al.*, 1995)

$$(\phi, \pi) \to (\psi + B, \Pi). \tag{7}$$

The field ψ has mass $M = \sqrt{t}\cdot m$, B is a constant, the field condensate and Π is the conjugate momentum (ψ). Therefore, the Hamiltonian in Eq. (2) can be written in the form;

$$H = \bar{H}_0 + \bar{H}_I + \bar{H}_1 + E, \tag{8}$$

where

$$\bar{H}_0 = N_M\left(\frac{1}{2}(\Pi^2 + (\Delta\psi)^2)\right) + \frac{1}{2}N_M(m^2 + 3g(B^2 + \Delta)\psi^2),$$

$$\bar{H}_I = \frac{g}{4}N_M(\psi^4 + 4B\psi^3 - J\psi).$$

$\bar{H}_1$ can be found as

$$\bar{H}_1 = N_M(m^2 + g(B^2 + 3\Delta)B\psi), \tag{9}$$

and the field independent terms can be regrouped as

$$E = \frac{1}{2}\left(m^2 + \frac{12g\Delta}{4}\right)B^2 + \frac{g}{4}B^4 + \frac{1}{8\pi}(M^2 - m^2) + \frac{3g\Delta^2}{4} + \frac{1}{2}m^2\Delta - JB. \tag{10}$$

Taking $b^2 = 4\pi B^2$ and the dimensionless parameters $t = \frac{M^2}{m^2}$, $G = \frac{g}{2\pi m^2}$ and $K = 4\sqrt{\pi}\frac{J}{m^2}$, the corresponding vacuum energy density can be written as

$$E(b, t, G) = \frac{m^2}{8\pi}\left(-Kb + b^2 + \frac{G}{4}(b^4 - 6b^2\ln t + 3\ln^2 t) + t - 1 - \ln t\right) \tag{11}$$

The renormalization conditions are given by (Peskin and Schroeder, 1995)

$$\frac{\partial^n}{\partial b^n}E(b, t, G) = g_n, \tag{12}$$

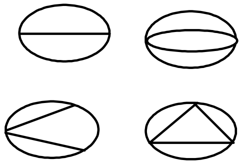

Fig. 1. The non-cactus Feynman diagrams due to the interaction Hamiltonian $\frac{g}{4}(\psi^4 + 4B\psi^3)$ up to g^3 order.

where g_n is the ψ^n coupling. For instance,

$$\frac{\partial E}{\partial B} = -J, \quad \frac{\partial^2 E}{\partial B^2} = M^2, \tag{13}$$

where $g_1 = -J$ and $g_2 = M^2 = m^2 + 3g(B^2 + \Delta)$. Note that, the renormalization condition $\frac{\partial E}{\partial B} = -J$ enforces $\bar{H}_1$ to be zero.

The normal ordered effective potential of $\frac{g}{4}\phi^4 - J\phi$ theory (Eq. (11)) possesses some remarkable features, such as manifest duality between symmetric ϕ^4 theory with large coupling g and broken symmetry ϕ^4 theory with small coupling G as well as very close value of the critical coupling $G_c = 1.625$ (Dineykhan *et al.*, 1995) to that obtained from the lattice calculation (Loinaz and Willy, 1998). Besides, for $K = 0$, it agrees with GEP results (Stevenson, 1985) which in turn accounts not only for the leading order diagrams but also for all the cactus diagrams (Chang, 1975; Lu and Kim, 2000). In fact, the variational procedure used in GEP calculations is equivalent to the normal ordered effective potential with renormalization conditions. To show this, consider the effective potential in Eq. (11) (for $K = 0$). In Stevenson (1985), the effective potential is minimized through:

$$\frac{\partial E}{\partial t} = 1 - t + \frac{3}{2}G(b^2 - \ln t) = 0 \tag{14}$$

$$\frac{\partial E}{\partial b} = b\left(1 + \frac{1}{2}G(b^2 - 3\ln t)\right) = 0, \tag{15}$$

which are exactly the equations obtained from the renormalization conditions $((\frac{\partial E}{\partial t} = 0) \equiv$ the mass renormalization condition $(\frac{\partial^2 E}{\partial b^2} = 2t))$.[3] Thus, the variational procedure in the GEP is equivalent to the normal ordered effective potential with the renormalization conditions. Accordingly, to go to higher orders in case of $K = 0$, we include only non-cactus diagrams (Fig. 1).

[3] In fact E here $\equiv \frac{8\pi E}{m^2}$.

In spite of all of the above correct features, the normal ordered effective potential in Eq. (11) describes a first order phase transition in contradiction with universality results. In order to improve the representation of the effective potential near the critical region, we consider the modification of Eq. (11) resulting from the higher order perturbative corrections to the vacuum energy followed by a Borel summation. In this work, we accounted for perturbative corrections up to G^3 order.

3. PERTURBATIONS AND BOREL RESUMMATION OF THE EFFECTIVE POTENTIAL

In the quasi-particle effective theory we have the interaction Hamiltonian in the form

$$\frac{g}{4}\left(\psi^4 + 4B\psi^3) - J\psi\right). \tag{16}$$

Accordingly, we have the Feynman diagrams shown in Figs. 1 and 2. The diagrams in Fig. 1 are calculated in Shalaby *et al.* (2005) and we give here the key points of the calculation procedure.

For a general ϕ^4 Hamiltonian,

$$\mathcal{H} = \frac{1}{2}(\partial\varphi)^2 + \frac{M^2}{2}\varphi^2 + a\varphi + h\varphi^3 + \frac{g}{4}\varphi^4, \tag{17}$$

the vacuum energy diagrams that contribute to $O(G^2)$ and $O(G^3)$ orders are shown in Figs. 1 and 2. These contributions are calculated using perturbation theory. For each of the Feynman diagrams, convergent integrals of the form

$$\mathcal{M} = C\int dk_1 \ldots dk_n \prod \frac{i}{k_i^2 - m^2 + i\epsilon} \prod \gamma_j \delta\left(\sum_i^{(j)} k_i\right) \tag{18}$$

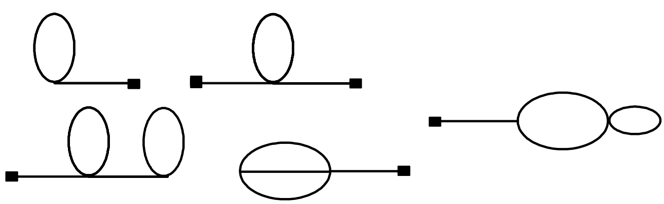

Fig. 2. The Feynman diagrams due to the presence of the external magnetic field. Here, the small square refers to the vertex $-J$. Note that we did not plot the tree level diagram but include it in the calculations.

were transformed to dimensionless variables $k_i \rightarrow m z_i$ and reduced to a form

$$\mathcal{M} = Cf \int dz_1 \ldots dz_n \prod \frac{1}{z_i^2 - 1 + i\epsilon} \prod \delta\Big(\sum_i^{(j)} z_i\Big). \qquad (19)$$

Here, C includes necessary $(2\pi)^d$ as well as factors of combinatorics and $\gamma_j \delta(\sum^{(j)} k_i)$ represent relevant vertices of the diagrams in Figs. 1 and 2 along with appropriate couplings. Also, f is an overall factor including γ_j's and a power of m obtained after the transformation to dimensionless variables. The integral was computed numerically using Monte Carlo method when a straightforward integration was not possible.

Following this procedure, the perturbative corrections to the vacuum energy density are given by

$$\delta E = -\frac{a^2}{2M^2} - 0.04452\frac{h^2}{M^2} - 0.00318\frac{g^2}{M^2} + 0.01998\frac{h^2 g}{M^4} + 0.0006507\frac{g^3}{M^4} + \ldots \qquad (20)$$

In terms of the parameters G, b and t, the contribution of the diagrams Figs. 1 and 2 to the first order prediction of the effective potential in Eq. (11) yields the result

$$\frac{8\pi E(t,b,G)}{m^2} = t - \ln t + b^2 - 1 + G\left(\frac{1}{4}b^4 + \frac{3}{4}\ln^2 t - \frac{3}{2}b^2 \ln t\right) + G^2\left(-\frac{3.155}{t} - 3.515\frac{b^2}{t}\right) + G^3\left(\frac{4.057}{t^2} + 9.918\frac{b^2}{t^2}\right) + \begin{pmatrix} -Kb + \frac{3}{4}G\frac{K^2}{t^2}\ln t - \frac{1}{2}\frac{K^2}{t} - 3.515\,2G^2\frac{K}{t^2}b + \frac{3}{2}G\frac{K}{t}(\ln t)\,b \\ -\frac{9}{4}G^2\frac{K}{t^2}\left(\ln^2 t\right)b - \frac{9}{4}G^2\frac{K}{t^2}(\ln t)\,b \end{pmatrix}. \qquad (21)$$

Applying the normalization condition in Eq. (13) one get

$$-K = 2b - K + G\left(\frac{1}{2t}(2tb^3 - 6t(\ln t)\,b + 3K\ln t)\right) + G^2\left(-7.03\frac{b}{t} - \frac{3.515\,2K + 2.25K\ln t(1+\ln t)}{t^2}\right) + G^3\left(19.836\frac{b}{t^2}\right), \qquad (22)$$

and

$$2t = 2 + G(3b^2 - 3\ln t) + G^2\left(-\frac{7.03}{t}\right) + G^3\left(\frac{19.836}{t^2}\right). \qquad (23)$$

Up to the best of our knowledge, the effective potential of the $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ model has never been calculated up to this order in perturbation series. While this may be viewed as an improvement over the original first order result, we note that with G^3 corrections the effective potential reproduces a first order phase transition. One can find that in this approximation one obtains a first order transition near G_c (*http://www.lib.ncsu.edu/theses/available/etd-08122004-160155/*). Thus, the result obtained in Cea and Tedesco (1994) (they found a second order phase transition) is fortuitous and changes when the effective potential is computed up to G^3 corrections. It may be natural to anticipate that perturbative potential computed for instance in G^4 order may again give rise to a different kind of phase transition. Thus, computing perturbative corrections alone does not lead to a meaningful improvement in the description of the critical region. A non-perturbative tool needs to be employed here as we discuss in this work. For further analysis, we attempt to improve perturbative series given by Eq. (21) with the Borel resummation performed by following the algorithm originally due to Kleinert *et al.* (1996).

In the series for the effective potential given by

$$E(G) = \sum_k Z_k G^k, \tag{24}$$

we assume that the large order behavior of the series is known to be

$$Z_k \to (-1)^k \, k! k^\delta \sigma^k \left(\gamma_0 + \frac{\gamma_1}{k} + \frac{\gamma_2}{k^2} + \cdots\right), \text{ as } k \to \infty, \tag{25}$$

where $\delta, \sigma, \gamma_0, \gamma_1 \ldots$ are some constants. Also, we assume that the strong coupling behavior of $E(G)$ is known to be

$$E(G) \to c_s G^\alpha, \text{ as } G \to \infty, \tag{26}$$

where α is a constant. Following Kleinert, Thoms and Janke, we rewrite the series given by Eq. (24) in terms of functions $I_p(G)$ given by

$$I_p(G) = \int_0^\infty dt e^{-t} t^c H_p^c(Gt), \tag{27}$$

where H_p^c are constrained in such a way that $I_p(G)$ satisfies both conditions in Eqs. (25) and (26) automatically. Here, $c = \delta + \frac{3}{2}$. Then, $E(G)$ can be written as

$$E(G) = \sum_{p=0}^\infty a_p I_p(G). \tag{28}$$

According to Kleinert, Thoms and Janke, it is convenient to choose the basis functions $I_p(G)$ in the following form (Kleinert *et al.*, 1996)

$$I_p(G) = \left(\frac{4}{\sigma G}\right)^{c+p} \int_0^1 \frac{(1+w)\, w^{c+1}}{\Gamma(c+1)\,(1-w)^{2c+5}} \exp\left(-\frac{4w}{(1-w)^2\, \sigma G}\right) dw, \tag{29}$$

so that the expansion coefficients are given by

$$a_p = \sum_{k=0}^{p} \frac{Z_k}{(c+1)_k} \left(\frac{4}{\sigma}\right)^k \binom{p+k-1-2\alpha}{p-k}. \tag{30}$$

This provides a Borel resumed form of the original series given by Eq. (24).

According to Eq. (21), the coefficients Z_k in our case are given by

$$\begin{aligned}
Z_0 &= t - \ln t - \frac{1}{2}\frac{K^2}{t} - Kb + b^2 - 1,\\
Z_1 &= \frac{1}{4}b^4 + \frac{3}{4}\ln^2 t - \frac{3}{2}b^2 \ln t + \frac{3}{2}\frac{K}{t}(\ln t)b + \frac{3}{4}\frac{K^2}{t^2}\ln t,\\
Z_2 &= -\frac{3.155}{t} - 3.515\frac{b^2}{t} - 3.515\, 2\frac{K}{t^2}b - \frac{9}{4}\frac{K}{t^2}(\ln^2 t)b\\
&\quad - \frac{9}{4}\frac{K}{t^2}(\ln^2 t)b - \frac{9}{4}\frac{K}{t^2}(\ln t)b,\\
Z_3 &= \frac{4.057}{t^2} + 9.918\frac{b^2}{t^2}.
\end{aligned} \tag{31}$$

The parameters σ, α and c are argued in Shalaby *et al.* (2005); *http://www.lib.ncsu.edu/theses/available/etd-08122004-160155/*). However, we will obtain the true value of the parameter c for the first time later in this work. In fact, the parameter σ is obtained in Reference Zinn-Justin (1993) ($\sigma = 0.238659217$). For the parameter α, it is obtained from the strong coupling behavior which according to duality it takes the value 1. After performing the Borel resummation for effective potential and taking into account $\alpha = 1$, we find from Eqs. (29) and (30) that

$$8\pi E(b,G)/m^2 = \sum_p a_p I_p = \left(\frac{4}{\sigma G}\right)^{c+1} \int_0^1 dw \frac{(1+w)w^c e^{-\frac{4w}{(1-w)^2\sigma G}}}{\Gamma(c+1)(1-w)^{2c+5}}$$
$$\times \left[Z_0 + w\left(-2Z_0 + \frac{Z_1}{\frac{\Gamma(c+2)}{\Gamma(c+1)}}\left(\frac{4}{\sigma}\right)\right)\right.$$

$$
+ w^2 \left(Z_0 + \frac{Z_2}{\frac{\Gamma(c+3)}{\Gamma(c+1)}} \left(\frac{4}{\sigma}\right)^2 \right)
$$

$$
+ w^3 \left(2 \frac{Z_2}{\frac{\Gamma(c+3)}{\Gamma(c+1)}} \left(\frac{4}{\sigma}\right)^2 + \frac{Z_3}{\frac{\Gamma(c+4)}{\Gamma(c+1)}} \left(\frac{4}{\sigma}\right)^3 \right) \Bigg], \tag{32}
$$

where $Z_p = Z_p(t, b)$ is given by Eq. (32) and dependence on the quasi-particle mass parameter t should be eventually removed using the mass renormalization condition.

The renormalization conditions can now be applied to the resummed effective energy as

$$
\frac{\partial^n}{\partial b^n} E(t, b, G) = g_n. \tag{33}
$$

This gives

$$
-K = \sum_p a_p^{(1)} I_p, \quad 2\frac{M^2}{m^2} = a_p^{(2)} I_p \tag{34}
$$

where

$$
a_0^{(1)} = 2b - K, \tag{35}
$$

$$
a_1^{(1)} = -2a_0^1 + \left(\frac{1}{2t}(2tb^3 - 6t(\ln t) b + 3K \ln t) \right) \frac{4}{\sigma(c+1)}, \tag{36}
$$

$$
a_2^{(1)} = a_0^1 + \left(-7.03\frac{b}{t} - \frac{3.515\,2K + 2.25K \ln^2 t}{t^2} \right) \left(\frac{4}{\sigma}\right)^2 \frac{1}{(c+1)(c+2)}, \tag{37}
$$

$$
a_3^{(1)} = \left(\frac{4}{\sigma}\right)^2 \frac{2a_2^1}{(c+1)(c+2)} + \left(\frac{4}{\sigma}\right)^3 \frac{19.836\frac{b}{t^2}}{(c+1)(c+2)(c+3)}, \tag{38}
$$

and

$$
a_0^{(2)} = 2, \tag{39}
$$

$$
a_1^{(2)} = -2a_0^2 + (3b^2 - 3\ln t)\frac{4}{\sigma(c+1)}, \tag{40}
$$

$$
a_2^{(2)} = a_0^2 + \left(-\frac{7.03}{t}\right) \left(\frac{4}{\sigma}\right)^2 \frac{1}{(c+1)(c+2)}, \tag{41}
$$

$$
a_3^{(2)} = \left(\frac{4}{\sigma}\right)^2 \frac{2a_2^2}{(c+1)(c+2)} + \left(\frac{4}{\sigma}\right)^3 \frac{\frac{19.836}{t^2}}{(c+1)(c+2)(c+3)}. \tag{42}
$$

Now, assume that one resums the perturbative series for the renormalized coupling directly (Eq. (22) and Eq. (23). It is easy to realize that we get exactly the renormalized coupling obtained from the effective potential (Eq. (34)) provided that the parameters in the Borel summation are the same for both the effective potential and the n-point functions. This may be considered as a consistency requirement: the renormalized couplings should be the same either we summed the effective potential and then obtain the renormalized coupling through the general definition of the renormalization conditions or obtain the perturbative renormalized coupling and then resum. According to this consistency, the effective potential as well as the n-point functions have the same large order behavior and strong coupling behavior.

Parameter c is the only parameter left unconstrained in our procedure. We will show that it have the same value as that for the γ function. For that, consider the physical mass given by (perturbatively)

$$m^2 = \frac{\partial^2 E}{\partial b^2}|_{b=0} = 9.918\frac{G^3}{t^2} - 3.515\frac{G^2}{t} - 1.5G\ln t + 1.0. \tag{43}$$

The γ function is defined by (Collins, 1984; Dineykhan *et al.*, 1995)

$$-\frac{t}{m^2}\frac{dm^2}{dt} = -\frac{t}{m^2}\left(-\frac{G}{t^3}(19.836G^2 - 3.515Gt + 1.5t^2)\right). \tag{44}$$

By resumming the series in the numerator and in the denominator, we get

$$\gamma = -\frac{t}{m^2}\sum_p a_p^{(3)} I_p^{\alpha, b_0}, \tag{45}$$

where

$$a_p^{(3)} = \sum_{k=1}^{p} \frac{S_k}{(b_0+1)_k}\left(\frac{4}{\sigma}\right)^k \binom{p+k-1-2\alpha}{p-k}, \tag{46}$$

with

$$S_1 = \frac{-1.5}{t}, \quad S_2 = \frac{3.515t}{t^2}, \quad S_3 = \frac{-19.836}{t^3}. \tag{47}$$

The parameters α and $b_0 - \frac{3}{2}$ are the strong coupling and the large order parameters of the γ function. Here, duality predicts α to be 1 and b_0 is known to be 4.5 (Zinn-Justin, 1993).

If we try to obtain the γ function directly from the resummed effective potential (Eq. 32) we will have the same results provided that $b_0 = c$. Accordingly, we get the large order behavior of the effective potential from the known large order behavior of the γ function and as a consequence, we have a resummed formula for the effective potential which is free of parameters.

4. NUMERICAL RESULTS

As described above, Eq. (32) provides an expression that interpolates the effective potential between perturbative result given by Eq. (21) for G small and large. It also provides description of the effective potential in the critical region where perturbative expansion neither in S- nor BS-phase is good. We thus used Eq. (32) to study the behavior of effective potential in the critical region. We used a self-consistency condition

$$\frac{\partial^2 E(B, M, g)}{\partial B^2} = i D^{-1}(0) = M^2, \tag{48}$$

where $D(k)$ is the Feynman propagator, to fix quasi-particle mass t in Eq. (32).

To test our results, we plot the effective potential versus the vacuum condensate for $K = 0$ and for different G values (Fig. 3). We realize that the Borel resummed effective potential corrected the false result concerning the order of phase transition predicted by both the normal ordered result and the perturbative (up to G^3) result in Eq. (11) and Eq. (21), respectively. To go further toward more confidence about how successful is the resummed effective potential in Eq. (32), we plotted the effective potential for different values of the parameter K and for values of the G coupling close to the critical coupling G_c (Fig. 4). Here, we can see that the condensate is a positive and monotonic increasing function of K as predicted by constructive field theory. To make this point more clearer, the plot in

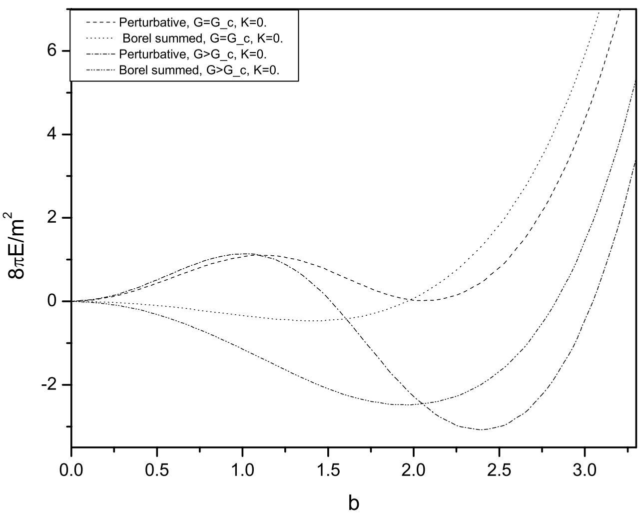

Fig. 3. The effective potential up to G^3 before and after Borel resummation near the critical coupling.

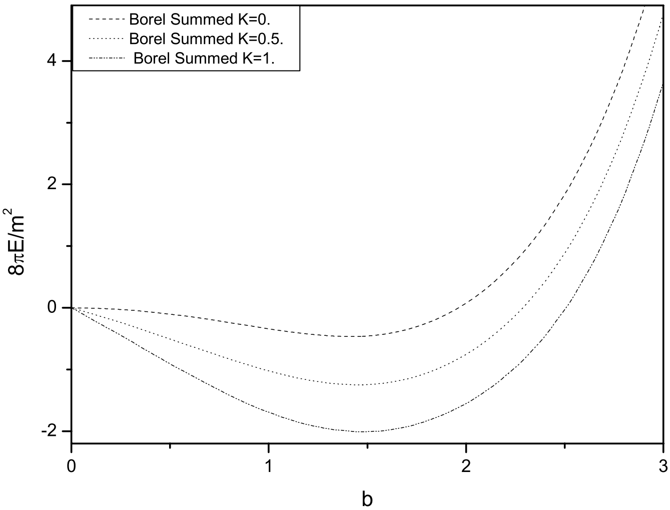

Fig. 4. The effective potential after Borel resummation near the critical coupling for different K values.

Fig. 5 is generated. The plot shows that $b \rightarrow 0$ as $K \rightarrow 0^+$ as well as showing that b is a positive monotonic function of K (not shown in the figure that the mass gap also agrees with features of the constructive field theory results). The final test we present here is to check the symmetry breaking in the presence of the external field K. Our knowledge about magnetic systems tells us that the symmetry is always broken in the presence of even a small external field. To check this for our calculations, we plotted the effective potential versus the condensate for different K values but this time for $G < G_c$ (Fig. 6). Rather than the perturbative result, the symmetry is always broken (no phase transition).

For the critical exponents, consider the graph in Fig. 7. The exponent ν_c can be extracted from the critical behavior of $E\alpha - \xi^{-d}$ (Kaku, 1993), where ξ is the correlation length (inverse of mass gap) and d is the space-time dimension. Since $\xi\ \alpha\ |K|^{-\nu_c}$ at the critical isotherm (Pelissetto and Vicari, 2002), we get the result $\nu_c = 0.58$ which is very close to the value found in the Ising model ($\nu_c = 0.53333$).

The exponent ν (Kaku, 1993) can be extracted from critical behavior of $E \sim (G - G_c)^{d\nu}$ (see Fig. 8), where d is the space-time dimension. This gives $\nu = 1$ which coincides with the value found in the Ising model. For the exponent γ we note that the perturbative result is

$$\gamma = \frac{-t}{m^2}\left(-\frac{G}{t^3}(19.836G^2 - 3.515Gt + 1.5t^2)\right), \tag{49}$$

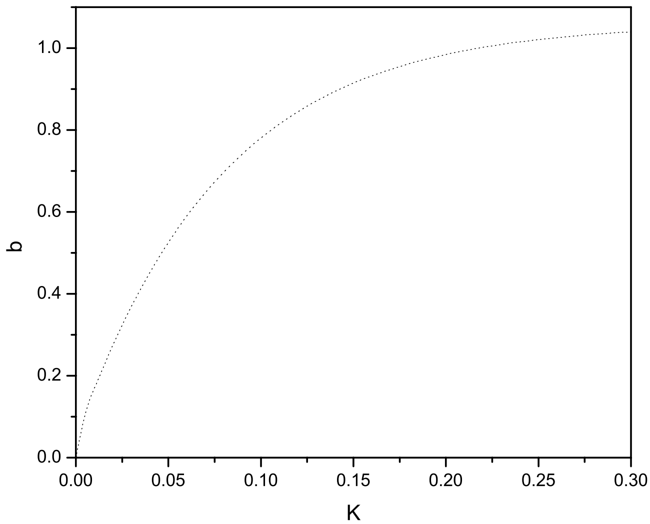

Fig. 5. The vacuum condensate b as a function of K at the critical isotherm.

which gives the mean field result $\gamma = 1$. The Borel resummed result is given by

$$\gamma = \frac{-t}{m^2}\left(\begin{array}{c} I_1\left(\frac{4}{t\sigma}\frac{-1.5}{c+1}\right) + I_2\left(\left(\frac{4}{t\sigma}\right)^2\frac{3.515}{(c+1)(c+2)}\right) \\ +I_3\left(\left(\frac{4}{t\sigma}\right)^2\frac{2(3.515)}{(c+1)(c+2)} + \left(\frac{4}{t\sigma}\right)^3\frac{-19.836}{(c+1)(c+2)(c+3)}\right)\end{array}\right), \tag{50}$$

where $m^2 = \frac{\partial^2 E}{\partial b^2}|_{b=0}$. Using the resummed effective potential in Eq. (32) we get the result

$$m^2 = I_0 + I_1\left(-2+\frac{4}{\sigma}\frac{-1.5\ln t}{c+1}\right)+I_2\left(1+\left(\frac{4}{\sigma}\right)^2\frac{-3.515}{t(c+1)(c+2)}\right) \tag{51}$$

$$+\,I_3\left(\left(\frac{4}{\sigma}\right)^2\frac{2(-3.515)}{t(c+1)(c+2)} + \left(\frac{4}{\sigma}\right)^3\frac{9.918}{t^2(c+1)(c+2)(c+3)}\right). \tag{52}$$

Our estimate for the exponent γ gives the result $\gamma = 1.667$ compared to the exact Ising result $\gamma = 1.75$.

According to the above results we feel that the algorithm we established is very successful in predicting all the known qualitative results from constructive field theory for the $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ field theory. For the quantitative predictions (exponents), we obtain good results compared with the exact Ising results. In fact,

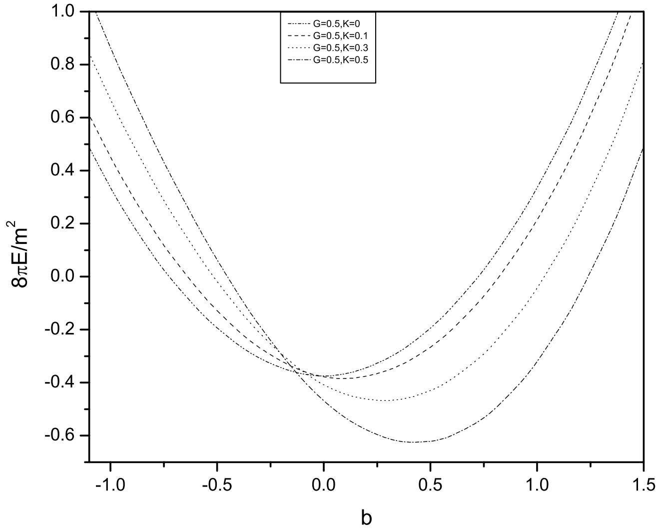

Fig. 6. The effective potential after Borel resummation at $G = 0.5$ for $K = 0.0, 0.1, 0.3, 0.5$.

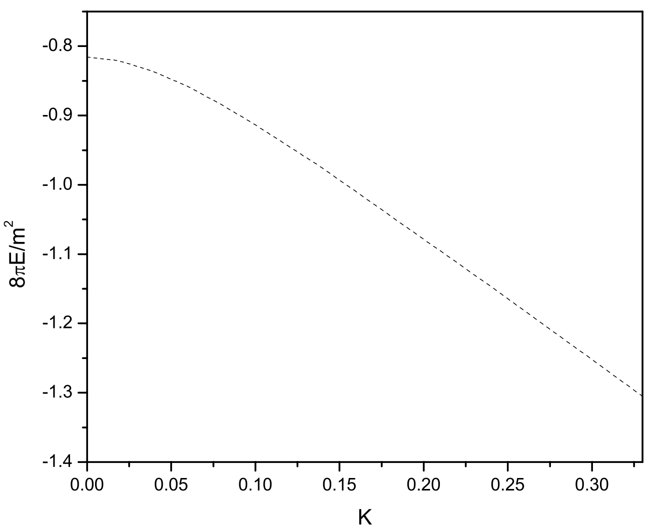

Fig. 7. The effective potential after Borel resummation at the critical isotherm $G = G_c$ as a function of K.

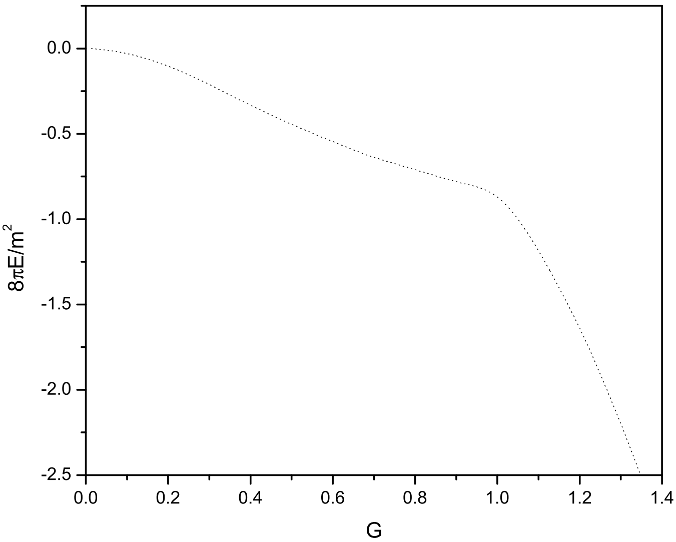

Fig. 8. The vacuum energy density (Borel resumed effective potential at its minimum) as function of coupling G for $K = 0$.

this is the first time to get such good results with the input perturbative series up to g^3 order for the $(\frac{g}{4}\phi^4 - J\phi)_{1+1}$ field theory. Accordingly, we guess that more refinement for the critical exponents can be obtained if we add one or two more terms to the perturbation series. However, such investigation will require substantial amount of time. It naturally becomes a topic of our future work to improve the current results.

5. CONCLUSION

In this paper we obtained a formula for the effective potential for ϕ^4 scalar field theory in $1 + 1$ dimensions that is applicable to all values of the coupling constant G. This formula is obtained by establishing a parameter-free Borel resummation of the perturbative effective potential and accounting for the strong coupling regime via duality related solutions. Our formula effectively interpolates between the two perturbative series valid, one for small G and the other for large G, in duality related representation. It is also applicable in the critical region while perturbative expansions in both original and duality related representations are not valid there.

To carry out Borel resummation, we calculated perturbatively the effective potential for ϕ^4_{1+1} scalar field theory up to the order of G^3 starting from the duality-related solutions. In Cea and Tedesco (1994), second order phase transition was observed with perturbative expansion up to the order G^2. However, up to the order G^3, we found that the perturbation series yields first order phase transition. This indicates that the result of Cea and Tedesco (1994) is fortuitous.

We improved our perturbative result by using the Borel resummation following algorithm suggested by Kleinert *et al.* (1996). This improvement resulted in the correct order of the phase transition as well as the agreement with the constructive field theory predictions.

We were able to obtain the large order behavior of the effective potential by obtaining the γ function in two different ways and enforce the parameters in the two formulae to match.

For the critical exponents we get a very close results compared to the exact Ising values although the input perturbative series we obtained is the shortest one ever been used for Borel resummation. Accordingly, we guess that more refinement for the critical exponents can be obtained if we add one or two more terms to the perturbation series.

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